K. SAITO'S CONJECTURE FOR NONNEGATIVE ETA PRODUCTS

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Abstract. We prove that the Fourier coefficients of a certain general eta product considered by K. Saito are nonnegative. The proof is elementary and depends on a multidimensional theta function identity. The $z = 1$ case is an identity for the generating function for p-cores due to Klyachko [13] and Garvan, Kim and Stanton [7].

1. Introduction

Throughout this paper $q = \exp(2\pi i\tau)$ with $\Im\tau > 0$ so that $|q| < 1$. As usual the Dedekind eta function is defined as

$$\eta(\tau) := \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau)) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{1.1}$$

An eta product is a finite product of the form

$$\prod_k \eta(k\tau)^{e(k)}, \tag{1.2}$$

where the $e(k)$ are integers. K. Saito [15] considered eta products that are connected with elliptic root systems and considered the problem of determining when all the Fourier coefficients of such eta products are nonnegative. Subsequent work contains the following

Conjecture 1.1. (K. Saito [17]) *Let N be a positive integer. The eta product*

$$S_N(\tau) := \frac{\eta(N\tau)^{\phi(N)}}{\prod_{d|N}\eta(d\tau)^{\mu(d)}} \tag{1.3}$$

has nonnegative Fourier coefficients.

The conjecture has been proved for $N = 2, 3, 4, 5, 6, 7, 10$ by K. Saito [15], [16], [17], [18], [19], for prime powers $N = p^\alpha$ by T. Ibukiyama [11], and for $\gcd(N, 6) > 1$ by K. Saito and S. Yasuda [20], who also showed that for general N, the coefficient of q^n in $S_N(\tau)$ is nonnegative for sufficiently large n. We prove the conjecture for general N.

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The case $N = p$ (prime) occurs in the study of p-cores. A partition is a p-core if it has no hooks of length p [7], [12]. p-cores are important in the study of p-modular representations of the symmetric group S_n. Define

$$E(q) := \prod_{n=1}^{\infty}(1-q^n), \tag{1.4}$$

and let $a_t(n)$ denote the number of partitions of n that are t-cores. It is well known that for any positive integer t

$$\sum_{n\geq 0} a_t(n)q^n = \frac{E(q^t)^t}{E(q)}. \tag{1.5}$$

This result is originally due to Littlewood [14]. See [7] for a combinatorial proof. Thus (1.5) implies that Conjecture 1.1 holds for $N = p$ prime since

$$S_p(\tau) = \frac{\eta(p\tau)^p}{\eta(\tau)} = q^{(p^2-1)/24}\frac{E(q^p)^p}{E(q)}. \tag{1.6}$$

Granville and Ono [9] have proved that $a_t(n) > 0$ for all $t \geq 4$ and all n. We also need the following identity due to Klyachko [13]

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\cdot\vec{1}_t=0}} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}} = \frac{E(q^t)^t}{E(q)}, \tag{1.7}$$

where $\vec{1}_t = (1, 1, \dots, 1) \in \mathbb{Z}^t$, $\vec{b}_t = (0, 1, 2, \dots, t-1)$, and t is any positive integer. See [7] for a combinatorial proof. See also [8, Prop.1.29] and [5, §2]. Our proof of K. Saito's Conjecture depends on the following extension of (1.7).

Theorem 1.2. *Let $a \geq 2$ be an integer. Then for $z \neq 0$ and $|q| < 1$ we have*

(1.8)

$$\begin{aligned} C_a(z;q) :=& \\ &\sum_{\substack{\vec{n}=(n_0,n_1,\dots,n_{a-1})\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} q^{\frac{a}{2}\vec{n}\cdot\vec{n}+\vec{b}_a\cdot\vec{n}}\left(z^{an_1+1}+z^{an_2+2}+\cdots+z^{an_{a-1}+a-1}+z^{-an_{a-1}}\right)\\ =& E(q)E(q^a)^{a-2}\prod_{n=1}^{\infty}\frac{(1-z^aq^{a(n-1)})(1-z^{-a}q^{an})}{(1-zq^{n-1})(1-z^{-1}q^n)} \end{aligned}$$

We note that (1.7) follows from (1.8) by letting $z \to 1$. The case $a = 3$ is equivalent to [10, (1.23)]. See [4, §3.3]. The case $a = 2$ can be written as

$$\sum_{n\in\mathbb{Z}} q^{2n^2+n}\left(z^{2n+1}+z^{-2n}\right) = \prod_{n=1}^{\infty}(1+zq^{(n-1)})(1+z^{-1}q^n)(1-q^n), \tag{1.9}$$

which follows easily from Jacobi's triple product identity [1, (2.2.10)].

Notation. We use the following notation for finite products

$$(z;q)_n = (z)_n = \begin{cases} \prod_{j=0}^{n-1}(1-zq^j), & n > 0\\ 1, & n = 0. \end{cases}$$

For infinite products we use

$$(z;q)_\infty = (z)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{n=1}^{\infty}(1-zq^{(n-1)}),$$

and

$$[z;q]_\infty = (z;q)_\infty(z^{-1}q;q)_\infty = \prod_{n=1}^{\infty}(1-zq^{(n-1)})(1-z^{-1}q^n),$$

for $|q|<1$ and $z\neq 0$.

2. Proof of Theorem 1.2

Suppose $a\geq 2$. The idea is to show both sides of (1.8) satisfy the same functional equation as $z\to zq$ and agree for enough values of z. Define

$$R_a(z;q) = E(q)E(q^a)^{a-2}\frac{[z^a;q^a]_\infty}{[z;q]_\infty}, \tag{2.1}$$

which is the right side of (1.8). An easy calculation gives

$$R_a(zq;q) = z^{-(a-1)}R_a(z;q). \tag{2.2}$$

We show that basically the a terms in the definition of $C_a(z;q)$ are permuted cyclically as $z\to zq$. To this end we define

$$Q_a(\vec{n}) = \frac{a}{2}\vec{n}\cdot\vec{n} + \vec{b}_a\cdot\vec{n}, \tag{2.3}$$

$$F_j(z;q) := \sum_{\substack{\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} z^{an_j+j}q^{Q_a(\vec{n})} \qquad (1\leq j\leq a-1), \tag{2.4}$$

and

$$F_0(z;q) := \sum_{\substack{\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} z^{-an_{a-1}}q^{Q_a(\vec{n})}. \tag{2.5}$$

Now suppose $2\leq j\leq a-2$. Let $\vec{e}_0=(1,0,\ldots,0)$, $\vec{e}_1=(0,1,\ldots,0)$, $\ldots$, $\vec{e}_{a-1}=(0,0,\ldots,0,1)$ be the standard unit vectors, $\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a$, and $\vec{n}\,'=(n_1,n_2,\ldots,n_{a-1},n_0)+\vec{e}_{j-1}-\vec{e}_{a-1}$. An easy calculation gives

$$Q_a(\vec{n}\,') - Q_a(\vec{n}) = an_j + j - \vec{n}\cdot\vec{1}_a. \tag{2.6}$$

Hence

$$\begin{aligned} F_{j-1}(z;q) &= \sum_{\substack{\vec{n}\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} z^{an_{j-1}+(j-1)}q^{Q_a(\vec{n})} \\ &= \sum_{\substack{\vec{n}\,'\in\mathbb{Z}^a\\ \vec{n}\,'\cdot\vec{1}_a=0}} z^{a(n_j+1)+(j-1)}q^{Q_a(\vec{n}\,')} \\ &= \sum_{\substack{\vec{n}\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} z^{an_j+j+(a-1)}q^{Q_a(\vec{n})+an_j+j} \\ &= z^{(a-1)}F_j(zq;q), \end{aligned} \tag{2.7}$$

and

$$F_j(zq;q) = z^{-(a-1)} F_{j-1}(z;q). \tag{2.8}$$

Similarly we find that

$$F_0(zq;q) = z^{-(a-1)} F_{a-1}(z;q), \tag{2.9}$$

by using the result that

$$Q_a(\vec{n}\,') - Q_a(\vec{n}) = -a n_{a-1} - (a-2)\vec{n}\cdot\vec{1}_a, \tag{2.10}$$

where $\vec{n}\,' = (-n_{a-2}, -n_{a-3}, \ldots, -n_1, -n_0, -n_{a-1})$. Also we have

$$F_1(zq;q) = z^{-(a-1)} F_0(z;q), \tag{2.11}$$

by using the result that

$$Q_a(\vec{n}\,') - Q_a(\vec{n}) = a n_1 + 1 - a\vec{n}\cdot\vec{1}_a, \tag{2.12}$$

where $\vec{n}\,' = (-n_0, -n_{a-1}, -n_{a-2}, \ldots, -n_2, -n_1) + \vec{e}_0 - \vec{e}_{a-1}$.

Since

$$C_a(z;q) = \sum_{j=0}^{a-1} F_j(z;q), \tag{2.13}$$

we have

$$C_a(zq;q) = z^{-(a-1)} C_a(z;q). \tag{2.14}$$

In view of [3, Lemma 2] or [10, Lemma 1] it suffices to show that (1.8) holds for a distinct values of z with $|q| < |z| \le 1$. It is clear that

$$C_a(z;q) = R_a(z;q) = 0, \tag{2.15}$$

for $z = \exp(2\pi i k/a)$ for $1 \le k \le a-1$. Finally, (1.8) holds for $z = 1$ since

$$C_a(1;q) = a \sum_{\substack{\vec{n}\in\mathbb{Z}^a \\ \vec{n}\cdot\vec{1}_a = 0}} q^{\frac{a}{2}\vec{n}\cdot\vec{n} + \vec{b}_a\cdot\vec{n}} = a\frac{E(q^a)^a}{E(q)} = R_a(1;q), \tag{2.16}$$

by (1.7) with $t = a$. This completes the proof of Theorem 1.2.

3. Proof of K. Saito's Conjecture

First we show that

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{\substack{n\ge 1 \\ (n,M)=1}} (1-q^n), \tag{3.1}$$

for any positive integer M. Now

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{d|M}\prod_{m=1}^{\infty}(1-q^{dm})^{\mu(d)} = \prod_{n=1}^{\infty}(1-q^n)^{\varepsilon(n)}, \tag{3.2}$$

where

$$\varepsilon(n) = \sum_{d|M \,\&\, d|n} \mu(d) = \sum_{d|(M,n)} \mu(d) = \begin{cases} 1 & \text{if } (M,n)=1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

by a well known property of the Möbius function, and we have (3.1).

For any positive integer N we define

$$S_N(q) := \frac{E(q^N)^{\phi(N)}}{\prod_{d|N} E(q^d)^{\mu(d)}}. \tag{3.4}$$

We wish to show that all coefficients in the q-expansion of $S_N(q)$ are nonnegative. We consider three cases.

Case 1. $N = p^\alpha$ where p is prime. This case was proved by Ibukiyama [11]. Alternatively, the case $\alpha = 1$ follows from (1.5) and then use an easy induction on α.

Case 2. $N = p\,M$, where p is prime, M is odd and $p \nmid M$. We have

$$\prod_{d|N} E(q^d)^{\mu(d)} = \prod_{d|M} E(q^d)^{\mu(d)} E(q^{pd})^{\mu(pd)} = \prod_{d|M} \left(\frac{E(q^d)}{E(q^{pd})}\right)^{\mu(d)}. \tag{3.5}$$

By (3.1) we have

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{\substack{n\ge 1\\(n,M)=1}} (1-q^n) = \prod_{n\ge 0} \prod_{\substack{(r,M)=1\\1\le r\le M-1}} (1-q^{Mn+r}) = \prod_{\substack{(r,M)=1\\1\le r\le \frac{M-1}{2}}} [q^r;q^M]_\infty. \tag{3.6}$$

Now for a a positive integer, $|q|<1$ and $z \ne 0$ we let

$$D_a(z;q) := \frac{E(q^a)^a}{E(q)} C_a(z;q) = E(q^a)^{2a-2}\frac{[z^a;q^a]_\infty}{[z;q]_\infty}, \tag{3.7}$$

so that

$$\begin{aligned} \prod_{\substack{(r,M)=1\\1\le r\le \frac{M-1}{2}}} D_p(q^r;q^M) &= \left(E(q^{pM})^{2p-2}\right)^{\phi(M)/2} \prod_{\substack{(r,M)=1\\1\le r\le \frac{M-1}{2}}} \frac{[q^{pr};q^{pM}]_\infty}{[q^r;q^M]_\infty} \\ &= E(q^N)^{\phi(N)} \prod_{d|M} \frac{E(q^{pd})^{\mu(d)}}{E(q^d)^{\mu(d)}} \qquad \text{(by (3.6))} \\ &= \frac{E(q^N)^{\phi(N)}}{\prod_{d|N} E(q^d)^{\mu(d)}} \qquad \text{(by (3.5))} \\ &= S_N(q). \end{aligned} \tag{3.8}$$

K. Saito's Conjecture holds in this case since each $C_p(q^r;q^M)$ has nonnegative coefficients by Theorem 1.2, and $E(q^{pM})^p/E(q^M)$ has nonnegative coefficients by (1.5) so that each $D_p(q^r;q^M)$ has nonnegative coefficients.

Case 3. $N = p^\alpha\,M$, where p is prime, M is odd, $p \nmid M$, and $\alpha \ge 2$. We let $N' = p\,M$. It is clear that

$$\prod_{d|N} E(q^d)^{\mu(d)} = \prod_{d|N'} E(q^d)^{\mu(d)}. \tag{3.9}$$

Hence

$$S_N(q) = \frac{E(q^N)^{\phi(N)}}{E(q^{N'})^{\phi(N')}} S_{N'}(q) = \left(\frac{E(q^{p^{\alpha-1}N'})^{p^{\alpha-1}}}{E(q^{N'})}\right)^{(p-1)\phi(M)} S_{N'}(q). \tag{3.10}$$

Here $S_N(q)$ is the product of two terms. The second term $S_{N'}(q)$ has nonnegative coefficients from Case 2. The first term has nonnegative coefficients using (1.5)

with q replaced with $q^{N'}$ and $t = p^{\alpha-1}$. Thus K. Saito's Conjecture holds in this case.

4. Other Products with Nonnegative Coefficients

In this section we state a number results for coefficients of other infinite products. More detail will appear in a later version of this paper. For a formal power series

$$F(q) := \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Z}[[q]]$$

we write

$$F(q) \succeq 0,$$

if $a_n \geq 0$ for all $n \geq 0$. For a formal power series $F(z_1, z_2, \ldots, z_n; q)$ in more than one variable we interpret $F(z_1, z_2, \ldots, z_n; q) \succeq 0$ in the natural way. The following result follows from the q-binomial theorem [1, Thm2.1].

Proposition 4.1. *If $|q|$, $|t| < 1$ then*

$$\frac{(at;q)_\infty}{(a;q)_\infty (t;q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(aq^n;q)_\infty (q)_n} \succeq 0. \tag{4.1}$$

Corollary 4.2. (i) *If a, b, $M \geq 1$ then*

$$\prod_{n=0}^{\infty} \frac{(1-q^{Mn+a+b})}{(1-q^{Mn+a})(1-q^{Mn+b})} \succeq 0. \tag{4.2}$$

(ii) *If m, $n > 1$ then*

$$\frac{E(q^m)E(q^n)}{E(q)} \succeq 0. \tag{4.3}$$

(iii) *If m, $n > 1$ and not both 2 then*

$$\frac{E(q^m)E(q^n)E(q^{mn})}{E(q)} \succeq 0. \tag{4.4}$$

Proposition 4.1 has a finite analogue. For $0 \leq m \leq n$ the Gaussian polynomial [1, p.33] is defined by

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = \frac{(q)_{m+n}}{(q)_n (q)_m} = \frac{(1-q^{n+1})\cdots(1-q^{n+m})}{(q)_m} \tag{4.5}$$

Since it is is the generating function for partitions with at most m parts each $\leq n$ it is a polynomial (in q) with positive integer coefficients. We have

Proposition 4.3. *If $L \geq 0$ then*

$$\frac{(z_1 z_2;q)_L}{(z_1;q)_L (z_2;q)_L} = \sum_{j=0}^{L} \begin{bmatrix} L \\ j \end{bmatrix}_q \frac{z_1^j}{(z_1 q^{L-j};q)_j (z_2 q^j;q)_{L-j}} \succeq 0. \tag{4.6}$$

The case $t = a^{-1}$ of Proposition 4.1 is related to the crank of partitions [2]. Let $M(m, n)$ denote the number of partitions of n with crank m. Then

$$(4.7) \qquad (1-z)\frac{E(q)}{[z;q]_\infty} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$
$$= 1 + (z + z^{-1} - 1)q + \sum_{n\geq 2} \left(\sum_{m=-n}^{n} M(m,n) z^m \right) q^n.$$

We note the coefficients on the right side of (4.7) are nonnegative except for the coefficient of $z^0 q^1$. By observing that

$$(1+z)(z + z^{-1} - 1) = z^2 + z^{-1}$$

we have

Proposition 4.4. *If $|q| < 1$ and $z \neq 0$ then*

$$(4.8) \qquad (1-z^2)\frac{E(q)}{[z;q]_\infty} = (1+z) \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \succeq 0.$$

Proposition 4.5. *If $|q| < 1$ and $z \neq 0$ then*

$$(4.9) \qquad \frac{E(q^2)[z^4;q^2]_\infty}{[z^2;q^2]_\infty [qz^3;q^2]_\infty} \succeq 0.$$

$$(4.10) \qquad \frac{E(q^3)(z^2;q^3)_\infty}{(q^3 z^{-1};q^3)_\infty (z;q)_\infty)} \succeq 0.$$

We make the following

Conjecture 4.6. *Suppose $|q| < 1$ and $z \neq 0$.*

(i) *If $p \geq 1$ then*

$$(4.11) \qquad \frac{E(q)}{(z;q)_\infty (qz^{-p};q)_\infty} \succeq 0.$$

(ii) *If a, b, m, $n \geq 1$ then*

$$(4.12) \qquad \frac{E(q^{ma+nb})}{(q^a;q^{ma+nb})_\infty (q^b;q^{ma+nb})_\infty} \succeq 0.$$

(iii) *For $a \geq 1$*

$$(4.13) \qquad \frac{[z^a;q]_\infty E(q)}{[z;q]_\infty [z^{a+1};q]_\infty} \succeq 0.$$

The case $p = 1$ of (4.11), the case $m = n = 1$ of (4.12) and the case $a = 1$ of (4.11) are all special cases of Proposition 4.1. The case $a = 2$ of (4.13) follows from Proposition 4.1 together the following identity due to Ekin [6, (42)]

$$(4.14) \qquad \frac{[z^2;q]_\infty E(q)}{[z;q]_\infty [z^3;q]_\infty} = \frac{E(q^3)}{[z^3;q^3]_\infty [z^{-3}q;q^3]_\infty} + z\frac{E(q^3)}{[z^3;q^3]_\infty [z^{-3}q^2;q^3]_\infty}$$

This identity was used by Ekin to prove an number of inequalities for the crank of partitions mod 7 and 11.

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